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μ-model for the statics of dry granular medium

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We propose the description of the granular matter which is based on distribution of dry friction coefficients. Using such a concept and a simple one-dimensional packing of grains we solve the silo problem. The friction coefficients at contacts are determined both by geometry of packing configuration and the stress distribution in a medium. Within such an approach the Janssen coefficient k_J is determined and its dependence on the particle-particle and boundary-particle friction coefficients is obtained. Also we investigate the conditions for the appearance of the maximum in the pressure distribution with the depth with overweight on top. As an outcome of our work we propose the general framework to the description of the granular matter as a continual medium which is characterized by the field of the dry friction tensor.

Introduction

The properties of granular matter differ drastically from those of other continuous media like solids, liquids and gases. Despite its mechanical nature the problem of description of granular media is still open problem. Some arguments put forward recently [1, 2] even question the possibility of the description of such media basing on the hydrodynamic approach. The latter usually applied to continuous media with short-range interparticle forces. But nonconservative nature of the friction force hampers direct application of such an approach based on local balance equations for physical quantities like momentum, energy, entropy, etc. In such a complex situation with the dynamics the static properties of the granular medium are easier to investigate. There are several characteristic static phenomena, which any theory should explain. They are connected with the static distribution of the pressure along a silo. Namely, the deviation from Pascal law and the nonmonotonic dependence of the apparent mass on the overweight on the top of a silo. Lets us review them shortly.

In contrast with conventional fluids, "hydrostatic" pressure in granular media reaches the asymptotic finite value p_0 at some finite depth λ, and can be described as

$$p(z) = p_0 \left(1 - \exp(-z/\lambda)\right) , \quad \lambda = \frac{R}{2\mu k_J} \tag{1}$$

where R is the width of the silo, μ is the static friction coefficient between the grains and the walls of the silo, and k_J is the so-called Janssen coefficient. First explanation of this behavior originated from the Jannsen model [3] (see also [4]), based on two simple assumptions: i) the linear relation between σ_{xz} and σ_{xx} components of the stress tensor, which is analogous to the Coulomb-Amonton law for the dry friction, and ii) the linear ratio between other two components of the stress tensor: $\sigma_{xx} = k_J \sigma_{zz}$. Since then, many approaches to the statics of granular medium have been proposed either giving the grounds for the Janssen relations [5, 6], or building the theory without using them [7, 8, 9].

One of the simplest approach treats the granular media within the framework of linear isotropic elasticity. Indeed, the Janssen coefficient k_J might be expressed through the Poisson ratio of the media, and numerics recovers the Janssen relations between the components of the stress tensor [6]. Detailed comparisons with experimental results even allow to extract the relation between the Poisson ratio and the granular packing fraction [6]. However, such a picture is hardly adequate for the granular media of rigid particles. In addition, it is hard to control realization of the Coulomb threshold condition everywhere at the walls, which, obviously, influences the experimental results [10].

Other approaches do not exploit elastic nature of the grains, but use the Coulomb-Amonton law for the stress tensor components inside the media or at the walls [11]. To compete with indeterminancy of the problem, the granular media is assumed to be at the verge of Coulombic failure everywhere in the bulk. However, this assumption about fully mobilized friction seems to be too restrictive [4], since the frictional forces can be varied provided that the medium

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stays at rest. In order to simplify the problem which include randomness in the distribution of such coefficients, specific assumptions about the geometrical distribution of the grains should be made, e.g. within some lattice models [8, 9]. Such models give the results which are consistent with continuum theories for the average stresses.

A particularly interesting issue is the weight distribution in a silo with some overweight on top. Experiments show the maximum in the pressure distribution with the depth, in contradiction with predictions of the simple Janssen model [11]. Interestingly, this maximum appears almost at the same depth where pressure at the silo without the overweight changes in e times. Although some of mentioned approaches do not contradict experiment, but the physical reasons of this phenomenon are still unclear.

The aim of this paper is to propose the description of the granular media which incorporates an additional dry friction tensor field. In the simplest 1D geometry considered below it reduces to the distribution of the dry friction coefficients. We show that in this case it is possible to get the expression for the Janssen coefficient. Besides, we are able to explain the the weight distributions in experiments with overweight as a result of the inhomogeneity in the friction coefficient distribution. The existence of characteristic scale for such inhomogeneity leads to the parametric dependence of the relation "apparent mass - filled mass" on the ratio of two length scales characteristic inhomogeneity and the saturation length for the bulk pressure.

The structure of the paper is as follows. In Section I we introduce the μ-model and illustrate it on the simple 1D granular packing, which resembles the real granular packing in silo. The results and comparisons are presented in Section II of the paper. We discuss the force distribution with and without overweight for simple packing and also find how Janssen coefficient k_J can be evaluated from microscopic characteristics of the media. In Section III we discuss the grounds of μ-model using the results in Section II. Conclusions are given in Section IV.

I. μ-MODEL FOR SIMPLE 1D PACKING

The grains at contact are subjected to static friction force. The value of such a force is determined both by the geometry of the contact surfaces and the stress at the contact. The value of the friction force is between 0 and the maximum value. The last is described by the Coulomb-Amonton law for static friction. Exact value is defined by the condition to keep the grain at rest. Since a grain is in contact with its neighbors at several points the net force is the sum of reaction forces for each contact. Because of this, granular system cannot be described using deterministic approach. There are more unknown variables than the equations. The granular matter can implement different force configurations even if the packing configuration is the same.

Between two grains the reaction force acts. It is expedient to split it into two components: normal (normal reaction force) and tangential (friction forces). For each point of contact we can write

$$F_\tau = \mu \cdot F_n \,, \tag{2}$$

where coefficient μ is different for different contact point but it's value is in the interval $[0, \mu_f]$, where μ_f is the static friction coefficient and determines the static friction angle. Obviously these coefficients for each contact point together with the normal stress determine the force configuration of the system. In the continuous limit, the set of these coefficients transforms into the tensor field μ. Such a tensor field becomes additional characteristic of a granular medium.

To illustrate the idea we apply this model to simple granular packing shown in Fig. 1. It is a regular pseudo-1D packing of identical rigid spheres in the vertical chute with a proper width. We choose 1-D model in order get explicit analytical results. Though such models seem to be oversimplified they grasp the main feature of the granular systems, namely the jamming. Also they are widely used in modelling avalanches within framework of SOC [12]. In particular as is shown in [1] 1-D systems shows nontrivial dynamic which is due to nonpotential character of the interparticle interactions. Moreover, according to (2) the coefficient of friction μ is defined in a way irrespective on the dimension of the system. Thus it can be defined both for the grains and for the mesoscopic regions of the granular matter thus giving rise to the introduction of the μ-field.

Denote the normal force between grains i and $i+1$ by $P_{i,i+1}$, and between the grain i and the wall by N_i. Let us denote the friction coefficients for each "grain-grain" pair by $\mu_{i,i+1}$, and the ones for the "grain-wall'" pair by σ_i. The set of values $P_{i,i+1}$, N_i, $\mu_{i,i+1}$, and σ_i determines the force configuration of the system. The geometrical configuration of the system is represented by the angles between normal to the surface of contact of two grains and the vertical as $\alpha_{i,i+1}$. The force equilibrium for each grain can be written as

$$\begin{aligned} &N_i - P_{i-1,i}\vartheta_{i-1,i} - P_{i,i+1}\vartheta_{i,i+1} = 0 \\ &-m_i g + \sigma_i N_i - P_{i-1,i}\theta_{i-1,i} + P_{i,i+1}\theta_{i,i+1} = 0 \,. \end{aligned} \tag{3}$$

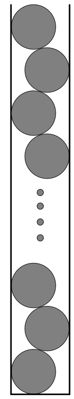

FIG. 1: Pseudo one-dimensional packing of identical spheres

where

$$\vartheta_{i,i+1} = \sin\alpha_{i,i+1} - \mu_{i,i+1}\cos\alpha_{i,i+1} \,,$$
$$\theta_{i,i+1} = \cos\alpha_{i,i+1} + \mu_{i,i+1}\sin\alpha_{i,i+1}$$

We omit the equation for the moment of force to illustrate how μ-approach works. The 1D nature of the system permits to write the expression for the the force between two particles, namely:

$$P_{i,i+1} = \frac{g}{\vartheta_{i,i+1}} \sum_{k=0}^{i} \frac{m_k T_{k+1,i}}{\eta_{k,k+1} + \sigma_k} + P_0 T_{0,i} \frac{\vartheta_{-1,0}}{\vartheta_{i,i+1}} \,, \tag{4}$$

where

$$T_{i_1,i_2} = \prod_{n=i_1}^{i_2} \frac{\eta_{n-1,n} - \sigma_n}{\eta_{n,n+1} + \sigma_n}$$

and

$$\eta_{i,i+1} = \frac{1 + \mu_{i,i+1} tg\alpha_{i,i+1}}{tg\alpha_{i,i+1} - \mu_{i,i+1}}$$

and P_0 is the overweight.

Equation (4) allows to analyze the load distribution through the silo with and without overweight with some distribution of the contact friction coefficients $\mu_{i,i+1}$.

II. RESULTS OF THE μ-MODEL FOR SIMPLE 1D PACKING

It is well known that force distribution in granular media depend not only on the silo height but also on the history and method of preparation of the granular matter sample [13, 14]. Within the approach proposed such a method can be modelled, e.g. by the distribution of the coefficients σ_i, which shows the stresses at the walls. Basing on the Eq. (4) as the exact microscopic solution of the model problem, we can model such a feature, by the set of coefficients $\{\mu_{i,i+1}, \sigma_i\}$ and angles $\{\alpha_{i,i+1}\}$. These data determine the geometrical and stress configuration of the system.

The question about deviation of static pressure distribution in granular media from the Pascal law is of particular interest since it is the characteristic feature of such kind of materials. Note that that Eq. (4) reduces to the Pascal law with the pressure being proportional to the depth if the walls are absolutely smooth, i.e. $\sigma_i = 0$. This limiting case is in correspondence with the fact that the deviation from the PL is due to nonlinear dependence of the tangential component of the stress along the wall.

Other configurations give the deviation form the PL. We investigate some of them numerically. To compare our results with known experimental data we take the random distributions of the friction coefficients $\sigma_i \in [0.25, 0.27]$ and $\mu_{i,i+1} \in [0.63, 0.73]$ with the limiting values chosen as the best fits to the simulational values of [6], and $\alpha_{i,i+1} = \pi/4$. Statistical averaging was performed over the 100 configurations for 50 grains, each one of mass 10 g. The resulting dependence between apparent and filled mass is shown in Fig. 2.

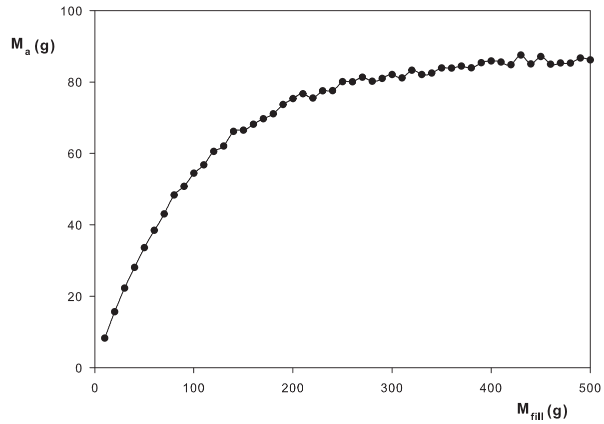

FIG. 2: Dependence the apparent mass on the filled mass of Eq. (4) with random distributions of $\mu_{i,i+1} \in [0.63, 0.73]$ and $\sigma_i \in [0.25, 0.27]$, and $\alpha_{i,i+1} = \pi/4$, $m_i = 10g$. Data are comparable with results of Ref. [6].

A. Force distribution in the silo without an overweight in continuous limit

Another relatively simple case is the "homogeneous" granular packing with $\mu_{i,i+1} = \mu$, $\sigma_i = \mu_w$ and $\alpha_{i,i+1} = \alpha$. In this case one can take the continuous limit in Eq. (4). For a silo without an overweight ($P_0 = 0$), one gets

$$P(\tilde{h}) = \rho g \lambda \left(1 - e^{-\tilde{h}/\lambda}\right) , \tag{5}$$

with characteristic length

$$\lambda = \frac{d}{\vartheta \cdot (\eta + \mu_w) \cdot \zeta \cdot \ln\left(\frac{\eta+\mu_w}{\eta-\mu_w}\right)} \tag{6}$$

where $\tilde{h}$ is the effective height: $\tilde{h} = h \cdot \vartheta \cdot (\mu_w + \eta)$, and configuration parameters

$$\eta = \frac{1 + \mu \cdot \tan\alpha}{\tan\alpha - \mu} , \quad \vartheta = \sin\alpha - \mu\cos\alpha , \quad \zeta = \frac{1 + \sin\alpha}{\cos\alpha}$$

and d is chute size.

So, we can see how the Janssen result can be obtained due to microscopic approach.

Note, that if $\tan\alpha = \mu$, Eq. (5) again reduces to the PL since grains do not lean against the wall. Another words the static friction between grains is enough to keep them at rest without any wall.

$$P(h) = \rho g h \cdot \cos\alpha . \tag{7}$$

Our result can be compared with Janssen formula (1). Formula (1) was obtained for vertical cylindrical chute. To compare it with results of our model it should be obtained for parallelepiped chute, which is infinite in one direction and has the profile as shown on Fig. 1. Calculation for this case transforms formula (1) to the following:

$$p(z) = p_0 \left(1 - \exp(-z/\lambda)\right) , \quad \lambda = \frac{d}{2\mu k_J} \tag{8}$$

In the "homogeneous" state the analogue of the Janssen coefficient k_J can be devised via comparison of Eqs. (5) and (6) with the result (8):

$$k_J = \frac{\vartheta \cdot \zeta}{2} \cdot (\eta/\mu_w + 1) \cdot \ln\left(\frac{\eta/\mu_w + 1}{\eta/\mu_w - 1}\right) . \tag{9}$$

Equation (9) relates the Janssen coefficient with the "microscopic" characteristics of the granular packing. In Fig. 3 we illustrate the dependence of k_J on friction at the wall (μ_w) for different values of internal friction μ.

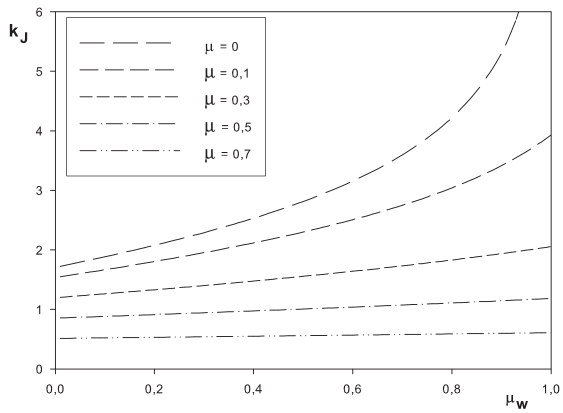

FIG. 3: The Janssen coefficient k_J as a function of friction μ_w on the walls from Eq. (9) for different μ.

B. Force distribution in the silo with overweight

The influence of the μ distribution on the distribution of the pressure can also be shown by considering the system when overweight is present.

For our numerical studies of the force distribution in the silo with overweight we use $\sigma_i = 0.25$ and value of overweight $P_0 = 80.8g$, which correspond to the value of grain-wall friction coefficient and overweight of Ref. [6]. Vessel contains 50 grains, each one of mass 10 g. The resulting dependence between apparent and filled mass is shown in Fig. 4. Bottom curve shows the force distribution in "homogeneous" media, where all $\mu_{i,\,i+1} = \mu_0$ without

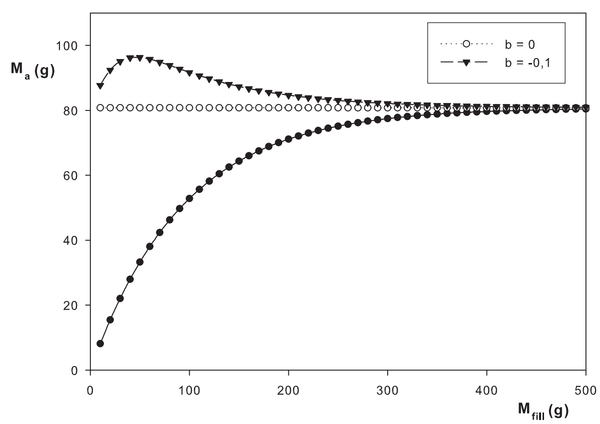

FIG. 4: Dependence of the apparent mass on the filled mass. Bottom curve: force distribution without overweight. Middle curve: force distribution with 80.8 g overweight for $\mu_{i,\,i+1} = 0.65$. Top curve: force distribution with 80.8 g overweight for $\mu_{i,\,i+1} = 0.65 - 0.1e^{-\,0.5\,i}$. For all curves $\sigma_i = 0.25$, $\alpha_{i,i+1} = \pi/4$, $m_i = 10(g)$. Data are comparable with results of Ref. [6].

overweight. As one can see from previous section this curve is nothing but Janssen exponential distribution. In case of overweight presence in such a "homogeneous" media, the force distribution is described by the middle curve. This result was also predicted by Janssen but it contradicts with the experiment. Since the force configuration is determined by the distribution of the coefficients $\mu_{i,\,i+1}$, we suppose that the latter has the same functional character as that for the pressure without overweight. The grounds of this assumption will be expanded in the next section.

Choosing the μ-distribution as following:

$$\mu_{i,\,i+1} = \mu_0 + be^{-\,c\,i}\,, \tag{10}$$

we adjust parameters b, c so that to achieve the best fit (the top curve on Fig. 4) to data of Ref. [6]. Thus we can see, that the maximum in the force distribution in the silo with overweight can be obtained if the friction coefficients are changed in the same way as the pressure changes without an overweight.

III. DISCUSSION OF THE μ-MODEL

The results obtained in previous section for simplified 1D model within the framework of μ-model can be extended further since the final results do not contain any microscopic characteristics of such a specific model. Indeed, the proposed approach allows us to switch the description of the granular media from the consideration of force network to the distribution of friction coefficients, which in continuous limit transforms to the field of tensor μ. Such a field becomes an additional characteristic of granular media which cannot be obtained due to common approach (e.g from Newton's equations). Additional statistical arguments about how this field is distributed should be used.

As one can see from Sec. II, we modelled the distribution of the μ in two different ways. First was uniform distribution $\mu = const$, and we showed how such an assumption conforms with previous theoretical results and experimental data. Another distribution was of exponential form and here we give the grounds for such a choice.

A. Spatial distribution of the μ-field

Let us consider the distribution of pressure in a silo with additional weight on its top. Suppose, that overweight is implemented by another silo of the same material, in which the pressure has reached its saturated value. Thus, the considered part of the granular media is actually belongs to the region where the pressure has reached it's saturated value. Therefore the pressure in this part must be equal to overweight. These reasonings are confirmed by Janssen's model [3] and our results, in which it is assumed that friction either at the wall or in the bulk is *const*. This is in obvious contradiction with the experiment [5], which shows that there is a maximum in the pressure dependence on depth.

It is possible to get such a nonmonotonic dependence of pressure on overweight in some theoretical approaches (see e.g. [7]), but there is no clear explanation why it appears. In addition, they are based on the assumptions $\mu = const$, which is adequate only near the Coulomb threshold. As we can see from the experiment, there are the "screening" region of size λ. In case of absence of the overweight, the pressure increases there and in case of overweight presence, the pressure has maximum within this region. Under overloading the stress configuration within this region changes more drastically than in the bulk. Within the approach proposed it can be described as the spatial distribution of the coefficients $\mu_{i,i+1}$ and σ_i, in general. In continuum approach it corresponds to some dependence $\mu(z)$ in the bulk and $\sigma(z)$ at the boundary.

Since μ characterizes the stress configuration, its spatial character must be similar to that of pressure in the system without overweight. This assumption can be viewed as a first approximation in the expansion spatially distributed function in a series of approximation. Indeed the function $\mu(z)$ can be written as

$$\mu(z) = \mu_0 + \mu_1(z) + \mu_2(z) + \dots \,. \tag{11}$$

Here μ_0 is constant part of μ, which is used in most of theoretical descriptions. Within the approach proposed $\mu_1(z)$ changes in the same way as the pressure without overweight changes. Then μ_2 takes into account changing in pressure in granular medium where $\mu(z) = \mu_0 + \mu_1(z)$, etc.

As one can see from previous section, to satisfy the experimental results, it is enough to model the distribution of $\mu_{i,i+1}$ as:

$$\mu_{i,i+1} = \mu_0 + be^{-ci}\,, \tag{12}$$

omitting higher order approximations. We take μ_0 as the bulk value, coefficient b itself governs the gap between bulk and border value of μ, both b and c are responsible for the speed of μ increasing. At the values of parameters, found in Sec. II, μ becomes saturated very quickly, so we have very thin region where μ changes from 0.55 to 0.65 (see Fig. 5).

But, as one can see from Fig 4, this region, where μ has such a dependence is enough to change the pressure distribution from the flat curve to the curve with the maximum. There is no maximum in the pressure dependence with depth, when $\mu = const$. This result, which was predicted by Janssen model, is confirmed by physical arguments, given above. Maximum appears when μ is distributed within a bulk in a given way. To satisfy the experimental results, region where μ changes in such a way must be narrow. Thus we can say about changing only near-boundary values of μ.

Note, that we have an essential difference in pressure dependencies only if overweight is present. When there is no overweight, the pressure dependence for constant μ and the pressure dependence for changing μ are almost the same. We can see that presence of an overweight reveals the real distribution of μ.

So we can conclude the following: a) the value of μ is not constant and has some distribution; b) such a distribution can be revealed only by the presence of an overweight and this distribution is essential only near the boundary.

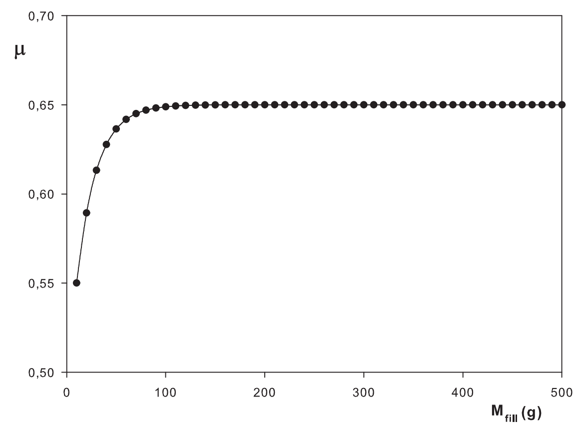

FIG. 5: Dependence of the spatially distributed μ on the depth in units of filled mass.

The last fact can be explained in the following way. When we fill the silo with the granular material some stress configuration is implemented in it. If then we put an overweight at the top of the silo, that configuration will become broken, and new one will be implemented. This happens because near-boundary layer feels this overweight and react on it. Other layers do not feel the overweight in essential way because of jamming in the upper layer. Other words, to fill the silo to height $2h$ at once is not the same that to fill the silo to height h first and then h again. This also illustrates how the stress distribution in the granular media depends on the history of packing creation.

B. Macroscopic parameters

Previous consideration allows to conclude that there must be at least two characteristic scales which characterize the state of silo under overloading. Followed by [6] we built rescaled dependencies (see Fig. 6).

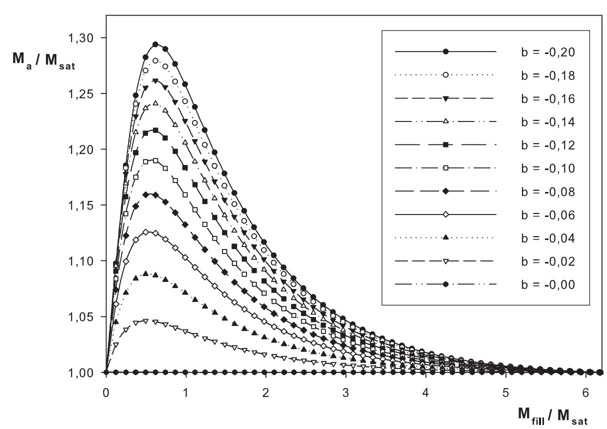

FIG. 6: Rescaled dependencies the apparent mass on the filled mass for different parameters b, which changes from -0.2 to 0 through 0.02. The bottom curve corresponds to $b = 0$, the upper one corresponds to $b = -0.2$. Overweight and saturated pressure equals 80.32 g. Apparent and filled mass expressed in overweight units. Other parameters are the same as for Fig. 4.

As one can see the bottom curve corresponds to the $\mu = const$ because the fact that $b = 0$ is equivalent to the $\mu_{i,i+1} = \mu_0$. For other values of b μ is not *const* and thus the apparent mass is not *const* either. The distribution of the apparent mass on filled mass has maximum, with value which depends on b shown at Fig. 7.

One can see from this that there is no universal rescaled curve for different parameters. This fact was also mentioned in [6]. Within the approach proposed this is the consequence of the existence of additional scale parameter λ_μ. Such a scale becomes evident in the presence of overweight if μ is not constant but μ is distributed within bulk. According

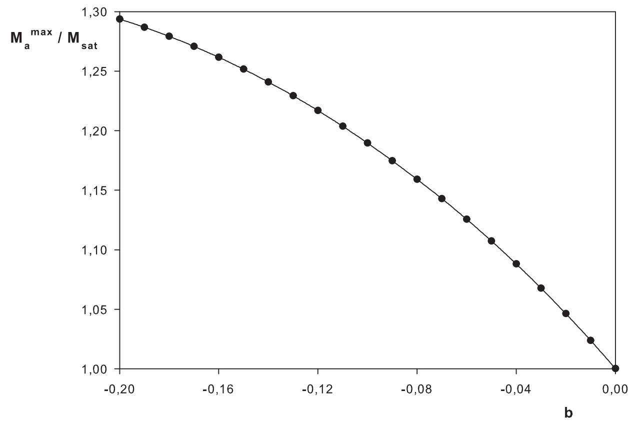

FIG. 7: Dependence the maximum of apparent mass on the μ changing gap in case of 80.83 g overweight presence.

to the proposed distribution of the μ, λ_μ depends on b and c, the gap in which μ changes and the steepness of μ saturation.

The dependencies shown on Fig. 6 must be parameterized using this parameter. In other words, this parameter splits Janssen's curve if silo is overloaded.

IV. CONCLUSION.

We propose the description of the granular media at rest based on the introduction of the spatial distribution for contact coefficients μ of dry friction. With the help of the simple static model we investigate the distribution of the weight in the silo. It is shown that in the case without overweight and homogeneous distribution of the friction coefficients the Janssen result is recovered. In a case of overweight we predict the maximum for the apparent mass as a function of a filled one, which is observed in experiments. It is important that the nature of such a maximum is related to the inhomogeneity in the spatial distribution of the dry friction coefficients. Such a distribution is formed due to the jamming of the grains in the upper layers which bear most of the overload. We put forward physical arguments which allow to obtain such distribution of the friction coefficients by taking into account the inhomogeneity of the pressure distribution with the height. Note the these result are obtained without any assumption about elasticity of the grains made in quasielastic approaches [6]. In addition within the proposed approach it is possible to explain the absence of simple rescaling law for the overshooting effect by the presence of two characteristic length. The first length is the Janssen length λ_J and characterizes the distribution of pressure, the other one is the scale of spatial distribution for dry friction coefficient.

In the continuous limit the development of the approach proposed implies the introduction of the tensor field for dry friction. Such a field becomes additional characteristic of a granular medium which is determined by both the geometrical and the stress configurations. The possibility of such a description is due to the fact that the interparticle interactions in the granular media are of short distance character [15]. It gives the grounds to expect that at least the static of granular medium should be described with the traditional framework of general elasticity theory with proper modifications.

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